Multiplicative noise: A mechanism leading to nonextensive statistical mechanics

Celia Anteneodo and Constantino Tsallis

Centro Brasileiro de Pesquisas Fisicas, Dr Xavier Sigaud 150, 22290-180, Rio de Janeiro-RJ, Brazil

A large variety of microscopic or mesoscopic models lead to generic results that accommodate naturally within Boltzmann-Gibbs statistical mechanics (based on $S_1 \equiv -k \int du\, p(u) \ln p(u)$). Similarly, other classes of models point toward nonextensive statistical mechanics (based on $S_q \equiv k[1 - \int du\, [p(u)]^q]/[q-1]$, where the value of the entropic index $q \in \Re$ depends on the specific model). We show here a family of models, with multiplicative noise, which belongs to the nonextensive class. More specifically, we consider Langevin equations of the type $\dot{u} = f(u) + g(u)\xi(t) + \eta(t)$, where $\xi(t)$ and $\eta(t)$ are independent zero-mean Gaussian white noises with respective amplitudes M and A. This leads to the Fokker-Planck equation $\partial_t P(u,t) = -\partial_u[f(u)P(u,t)] + M\partial_u\{g(u)\partial_u[g(u)P(u,t)]\} + A\partial_{uu}P(u,t)$. Whenever the deterministic drift is proportional to the noise induced one, i.e., $f(u) = -\tau g(u)g'(u)$, the stationary solution is shown to be $P(u,\infty) \propto \left\{1 - (1-q)\beta[g(u)]^2\right\}^{\frac{1}{1-q}}$ (with $q \equiv \frac{\tau+3M}{\tau+M}$ and $\beta = \frac{\tau+M}{2A}$). This distribution is precisely the one optimizing S_q with the constraint $\langle[g(u)]^2\rangle_q \equiv \{\int du\, [g(u)]^2[P(u)]^q\}/\{\int du\, [P(u)]^q\} =$ constant. We also introduce and discuss various characterizations of the width of the distributions.

Ubiquitous systems can be naturally described within Boltzmann-Gibbs (BG) statistical mechanics (based on the entropy $S_1 \equiv -k \int du\, p(u) \ln p(u)$). These systems have in common the fact that they like *equally* to live everywhere they are allowed to, that is, they are ergodic in the entire phase space. However, there are other systems that, depending on the initial conditions, may prefer a particular subspace. The BG scenario may not be appropriate any longer and an extension of the usual thermo-statistical description, taking into account the features of such subspace, would be required. If that subspace has a scale invariant geometry, a hierarchical or multi-fractal structure, then the model points toward nonextensive statistical mechanics (based on the entropic form $S_q \equiv k\,[1 - \int du\, [p(u)]^q]/[q-1]$, $q \in \Re$) [1] (see [2] for reviews). Among the models which belong to this category, one finds low-dimensional dissipative and conservative maps [3], fractional and nonlinear Fokker-Planck equations [4], Langevin dynamics with fluctuating temperature [5], growth of many-body scale-free networks [6] and long-range many-body classical Hamiltonians [7]. The corresponding value of the entropic index q depends on the specific model, or, more precisely, on the nonextensivity universality class of the model. It is our purpose here to show a large family of models with multiplicative noise which belongs to the nonextensive class.

Microscopic dynamics, containing multiplicative noise, may be encountered in many dynamical processes, such as in stochastic resonance [8], noise induced phase transitions [9], granular packings [10], and others [11,12]. Due to its significance, stochastic processes with multiplicative noise have been the subject of numerous studies in the last decades [13–15]. Here we will consider processes subject to both additive and multiplicative noises and described by the dimensionless stochastic differential equation of the form

$$\dot{u} = f(u) + g(u)\xi(t) + \eta(t), \qquad (1)$$

where $u(t)$ is a stochastic variable, f, g are arbitrary functions ($g(0) = 0$), and $\xi(t), \eta(t)$ are uncorrelated and Gaussian-distributed zero-mean white noises, hence satisfying

$$\langle\xi(t)\xi(t')\rangle = 2M\delta(t-t'), \quad \langle\eta(t)\eta(t')\rangle = 2A\delta(t-t'), \quad (2)$$

where $M, A > 0$ are the noise amplitudes and stand for "multiplicative" and "additive", respectively. Clearly, some degree of correlation between the two noises can be of physical relevance, however, this remains out of the present scope. The deterministic drift $f(u)$ can be interpreted either as a damping force (whenever u is a velocity-like quantity) or as an external force (when motion is overdamped and u represents a position coordinate). Other interpretations are possible as well, depending on the particular system treated.

It is interesting to note that additive and multiplicative terms in Eq. (1) could be gathered in an effective multiplicative noise term [16], however we prefer to keep track of both sources independently.

The stochastic differential equation (1) is not completely defined and must be complemented by an additional rule. This is due to the fact that each pulse of the stochastic noise produces a jump in u, then the question arises: which is the value of u to be used in $g(u)$? This is the well known Itô-Stratonovich controversy [15,17]. In the Itô definition, the value before the pulse must be used, whereas in the Stratonovich definition the values before and after the pulse contribute in a symmetric way. If noise were purely additive, then both definitions agree.

The Fokker-Planck equation for the probability density $P(u,t)$, associated to Eq. (1), can be obtained from the Kramers-Moyal expansion $\partial_t P = \sum_{n\geq 1}(-\partial_u)^n[D^{(n)}P]$, where the coefficients are given by $D^{(n)}(x,t) = \frac{1}{n!}\lim_{\tau\to 0}\frac{[u(t+\tau)-x]^n}{\tau}|_{u(t)=x}$. These coefficients can be readily obtained following the standard lines found for instance in [15]. Using the Stratonovich definition of stochastic integral, one gets

$$D^{(1)}(u,t) = f(u) + Mg(u)g'(u) \equiv J(u), \qquad (3)$$

$$D^{(2)}(u,t) = A + M[g(u)]^2 \equiv D(u), \qquad (4)$$

while $D^{(n)}(u,t) = 0$ for $n \geq 3$. Then, one arrives straightforwardly at

$$\partial_t P = -\partial_u j(u). \qquad (5)$$

where $j(u) \equiv J(u)P - \partial_u[D(u)P]$ is the current. In the Itô calculation, Eq. (3) becomes $J(u) = f(u)$, that is, the noise-induced (or spurious) drift is missing. In what follows we will adopt the Stratonovich definition. However, this choice will not affect the present discussion excepting for a redefinition of some of the parameters that are involved.

Eq. (5) can also be written as

$$\partial_t P = -\partial_u(f(u)P) + M\partial_u(g(u)\partial_u[g(u)P]) + A\partial_{uu}P. \qquad (6)$$

In some processes, the deterministic and noise-induced drifts may have the same functional form. Let us put this condition as follows:

$$f(u) = -\tau g(u)g'(u), \qquad (7)$$

τ being a proportionality constant. In other words, $f(u)$ is derived from a potential-like function $V(u) = \frac{\tau}{2}[g(u)]^2$. Let us note that the particular case $g(u) \propto f(u) \propto u$, which is a natural first choice for a physical system, verifies this condition. However, since no extra calculational difficulties emerge, we will discuss here the more general case (7). Notice that in the absence of deterministic forcing, condition (7) is trivially satisfied for any g by settig $\tau = 0$.

In the present paper we will restrict to the stationary solutions for no flux boundary conditions (i.e., such that $j(-\infty) = j(\infty) = j(u) = 0$, although more general conditions could in principle also be considered. If (7) is verified, then, the stationary solution $P_s(u)$ is of the q-exponential form appearing in nonextensive statistical mechanics [1]. More precisely, in that case, one obtains

$$P_s(u) \propto \left[1 + (q-1)\beta[g(u)]^2\right]^{\frac{1}{1-q}}, \qquad (8)$$

where $\beta \equiv \frac{1}{kT} = \frac{\tau+M}{2A}$ (where by T we generically mean the amplitude of an effective noise) and

$$q = \frac{\tau + 3M}{\tau + M}. \qquad (9)$$

As an aside comment let us mention that if we had used the Itô convention we would have obtained $q = \frac{\tau+4M}{\tau+2M}$. Let us now go back to our Stratonovich choice.

P_s is normalizable if $|g(u)|$ grows with $|u|$ faster than $|u|^{\frac{1}{1+\tau/M}}$ ($\tau > -M$). This probability distribution function (pdf) optimizes

$$S_q \equiv k\frac{1 - \int du \, [P(u)]^q}{q-1}, \qquad (10)$$

with the constraint

$$\langle[g(u)]^2\rangle_q \equiv \frac{\int du \, [g(u)]^2 \, [P(u)]^q}{\int du \, [P(u)]^q} = \text{ constant.} \qquad (11)$$

The condition (7) is not necessary for having solutions of the q-exponential form, as it follows along the lines of [18,19], in spite of the fact that the models therein considered are different from the present one.

Let us consider in more detail the case when both the forcing and the multiplicative noise depend on the stochastic variable u as a power law, that is, when the Langevin Eq. (1) becomes

$$\dot{u} = -\gamma u|u|^{r-1} + u|u|^{s-1}\xi(t) + \eta(t), \qquad (12)$$

with $r, s \geq 0$ and a drift coefficient γ typically positive. In this case, the deterministic drift is derived from a confining potential-like function of the form $V(u) = \gamma|u|^{r+1}/(r+1)$. The corresponding Fokker-Planck equation becomes

$$\partial_t P = -\partial_u(-\gamma u|u|^{r-1}P) + M\partial_u(|u|^s\partial_u[|u|^sP]) + A\partial_{uu}P. \qquad (13)$$

Hence, its stationary solution $P_s(u)$ is given by

$$P_s(u) = \frac{P_o \, e^{-h(u)}}{(1 + \frac{M}{A}|u|^{2s})^{1/2}}, \qquad (14)$$

P_o being the normalization constant and

$$h(u) \equiv \frac{\gamma|u|^{1+r}}{A[1+r]} \, {}_2F_1\Big(\frac{1+r}{2s}, 1; 1 + \frac{1+r}{2s}; -\frac{M}{A}|u|^{2s}\Big), \qquad (15)$$

where ${}_2F_1$ is the hypergeometric function.

We shall analyze now some limiting cases:

(A) For vanishing deterministic forcing ($\gamma \rightarrow 0$), Eq. (14) becomes

$$P_s(u) = \frac{P_o}{(1 + \frac{M}{A}|u|^{2s})^{1/2}}, \qquad (16)$$

with

$$P_o = \frac{(M/A)^{1/(2s)} \, \Gamma(\frac{1}{2})}{2\,\Gamma(1+\frac{1}{2s}) \, \Gamma(\frac{1}{2} - \frac{1}{2s})}, \qquad \text{for } s > 1. \qquad (17)$$

This pdf is of the q-exponential form (with $q = 3$) as expected because, in this case, condition (7) is trivially true since it corresponds to $\tau = 0$. Interestingly, in the presence of multiplicative noise, the steady state P_s is normalizable *even in the absence of a confining potential*, as long as $s > 1$. The so called spurious drift which originates from the multiplicative noise term in the Langevin

equation is responsible for kicking the system back close to the origin, where fluctuations are smaller. On the other hand, the additive noise plays an essential role, providing fluctuations which avoid the full concentration (at the origin) that would occur otherwise. This type of stabilizing effect of the multiplicative noise is long known [20].

(B) In the limit $M \to 0$, all other parameters being fixed, i.e., when no multipliciative noise is present, Eq. (14) becomes of the following stretched exponential form:

$$P_s(u) = \frac{\left[\frac{\gamma}{A[1+r]}\right]^{\frac{1}{1+r}}}{2\,\Gamma(1 + \frac{1}{1+r})}\, e^{-\frac{\gamma}{A[1+r]}\,|u|^{1+r}}. \tag{18}$$

In particular, for the linear forcing ($r = 1$), the Gaussian pdf is recovered.

(C) In the limit $A \to 0$, i.e., for vanishing strength of the additive noise, the steady state is normalizable for $s < 1$ and $p \equiv r + 1 - 2s > 0$. The condition $p > 0$ implies that the potential of the drift has to be steep enough to confine the system and yield a steady state. Then, we have

$$P_s(u) = \frac{p\left[\frac{\gamma}{Mp}\right]^{\frac{1-s}{p}}}{2\,\Gamma(\frac{1-s}{p})}\, \frac{e^{-\frac{\gamma}{Mp}\,|u|^p}}{|u|^s}. \tag{19}$$

Vanishing A concentrates the probability at the origin. Again, the Gaussian distribution is recovered for $r = 1$ (harmonic forcing) and $s = 0$ (additive noise).

The limits analyzed above are not generically interchangeable, that is, convergence is not necessarily uniform.

In the general case where (A, M, γ) are all finite, the same dependence on u as that in Eq. (19) is obtained for sufficiently large $|u|$. In fact, if one defines the dimensionless variable $\bar{u} = u/\lambda$ with $\lambda \equiv (A/M)^{\frac{1}{2s}}$, it is clear that the asymptotic expression of Eq. (14) for $|\bar{u}| \to \infty$ corresponds to both limits $|u| \to \infty$ and $A \to 0$.

For finite (A, M, γ), in the $p = 0$ marginal case where the drift and multiplicative-noise exponents are related through $1 + r = 2s$ (hence condition (7) is verified with $\tau = \gamma/s$), a q-exponential pdf emerges at asymptotically long time. In fact, in that case, the hypergeometric in Eq. (15) becomes $_2F_1(1, 1; 2; -z) = \ln(1 + z)/z$ and one gets

$$P_s(u) = \frac{P_o}{(1 + |u/\lambda|^{2s})^{\frac{1}{q-1}}}, \tag{20}$$

with

$$q = \frac{(\gamma/s) + 3M}{(\gamma/s) + M} \tag{21}$$

and

$$P_o = \frac{\Gamma(\frac{1}{2} + \frac{\gamma}{2sM})}{2\lambda\,\Gamma(1 + \frac{1}{2s})\,\Gamma(\frac{1}{2} + \frac{\gamma}{2sM} - \frac{1}{2s})}. \tag{22}$$

$P_s(u)$ is normalizable for $s + \gamma/M > 1$. Notice that q-exponential pdf's can appear even for negative γ (i.e., for repulsive deterministic forces). The $1 + r = 2s$ class includes the particular linear case $r = s = 1$, that has already been treated in the literature [11,21]. In Fig. 1 we exhibit the steady state pdf (20) for several values of the system parameters.

Assuming that $\langle u \rangle = 0$, the width of $P(u, t)$ can be characterized in many ways, such as (i) the inverse of the height at the origin, namely $1/P(0, t)$; (ii) the width at half height, Δ; (iii) the square root of the mean value of u^2; (iv) the $2s$-root of the mean value of $|u|^{2s}$; (v) the square root of the q-expectation values of u^2; (vi) the $2s$-root of the q-expectation value of $|u|^{2s}$, or even combinations of these, for instance (vii) $\sqrt{\Delta/P(0, t)}$ (see also [22]). Let us recall that the normalized q-expectation of ψ is defined as

$$\langle \psi \rangle_q(t) \equiv \frac{\int du\,\psi(u)[P(u, t)]^q}{\int du\,[P(u, t)]^q}, \tag{23}$$

hence, $\langle \psi \rangle_1$ equals the usual mean value $\langle \psi \rangle$.

For the usual case where $P(u, t)$ is a Gaussian, all these definitions basically coincide. This is not so in general, as we shall illustrate in what follows for the stationary state of Eq. (20).

(i) The inverse of the height at the origin:

$$\frac{1}{P_s(0)} = \lambda\,\frac{2\,\Gamma(1 + \frac{1}{2s})\,\Gamma(\frac{1}{2} + \frac{\gamma}{2sM} - \frac{1}{2s})}{\Gamma(\frac{1}{2} + \frac{\gamma}{2sM})}. \tag{24}$$

(ii) The width at half height:

$$\Delta = 2\lambda\,(2^{1/(\frac{\gamma}{2sM} + \frac{1}{2})} - 1)^{\frac{1}{2s}}. \tag{25}$$

(iii) The square root of the mean value of u^2:

$$\langle u^2 \rangle^{1/2} = \lambda\,\sqrt{\frac{\Gamma(\frac{3}{2s})\,\Gamma(\frac{1}{2} + \frac{\gamma}{2sM} - \frac{3}{2s})}{\Gamma(\frac{1}{2s})\,\Gamma(\frac{1}{2} + \frac{\gamma}{2sM} - \frac{1}{2s})}}, \tag{26}$$

for $\gamma/M > 3 - s$ (otherwise it diverges).

(iv) The $2s$-root of the mean value of $|u|^{2s}$,

$$\langle |u|^{2s} \rangle^{1/2s} = \frac{\lambda}{\left(\gamma/M - s - 1\right)^{\frac{1}{2s}}} \tag{27}$$

for $\gamma/M > 1 + s$ (otherwise it diverges).

(v) The square root of the q-expectation value of u^2:

$$\langle u^2 \rangle_q^{1/2} = \lambda\,\sqrt{\frac{\Gamma(\frac{3}{2s})\,\Gamma(\frac{3}{2} + \frac{\gamma}{2sM} - \frac{3}{2s})}{\Gamma(\frac{1}{2s})\,\Gamma(\frac{3}{2} + \frac{\gamma}{2sM} - \frac{1}{2s})}}, \tag{28}$$

for $\gamma/M > 3(1 - s)$ (otherwise it diverges), and

(vi) The $2s$-root of the q-expectation value of $|u|^{2s}$

$$\langle |u|^{2s} \rangle_q^{1/2s} = \frac{\lambda}{\left(\gamma/M + s - 1 \right)^{\frac{1}{2s}}}, \qquad (29)$$

for $\gamma/M > 1 - s$. Notice that this condition has already been encountered. Indeed, it is necessary for normalizability. An important remark is mandatory: of all the above characterizations of width which are based on expectation values (i.e., $\langle u^2 \rangle^{1/2}$, $\langle |u|^{2s} \rangle^{1/2s}$, $\langle u^2 \rangle_q^{1/2}$ and $\langle |u|^{2s} \rangle_q^{1/2s}$), *only the last one does not diverge in any admissible (i.e., normalizable) case.* This is particularly remarkable because this generalized expectation value is, as already seen, precisely the one to be used as constraint in the optimization of S_q. In addition to this, it is worthy noticing that $\langle |u|^{2s} \rangle_q^{1/2s}$ is in all cases comparable to $\sqrt{\Delta/P_s(0)}$, a semi-empirical quantity which takes into account the contributions of both body and tails.

In Fig. 2 we exhibit all these quantities as a function of M/γ for typical values of s with $r = 2s - 1$. As M/γ increases, in all cases, $P_s(u)$ becomes more peaky around the origin (Δ decreases). Moreover, for increasing M/γ and $s = 1$, the tail tends to $1/|u|$ (hence $P_s(0) \to 0$). Also for $s = 1/2$, $P_s(0) \to 0$ as the pdf becomes more tailed, but M/γ has the upper bound 2. Contrarily to the previous cases, for $s > 1$, as M/γ increases, $P_s(0)$ also increases.

The q-exponential character is not exclusive of the steady state but it also emerges along the time evolution of the pdf, as illustrated in Fig. 3. In this figure we employ the semi-$\ln_q$ representation, where $\ln_q x = (x^{1-q} - 1)/(q - 1)$. Notice in Figs. 3.b and 3.c that the curve for $t \to \infty$ is a straight line. In 3.c, curves are almost straight lines, thus indicating that the pdf's are very close to q-exponentials along time.

In the presence of multiplicative noise, the system variables directly couple to noise. Therefore, behaviors are observed that can not occur in the presence of additive noise alone. In particular, q-exponential pdf's of the form given by Eq. (8) can arise. For the present processes, q-exponentials do occur either in the absence of forcing ($\gamma = 0$) or for drifts verifying Eq. (7), which, for the power-law case, becomes $r = 2s - 1$. The deterministic forcing does not need to be confining (i.e., $\gamma < 0$ is allowed) for the formation of a q-exponential stationary pdf, provided $\gamma/M > 1 - s$. In any case, if a repulsive effective force prevails for some period of time, the variable u can take large values and power law tails can arise. In particular, a q-exponential occurs when the stochastic forcing is proportional to the deterministic one. Alternatively, during the intervals when the effective force is attractive, the probability tends to concentrate at the origin. Then, at this stage, the additive noise plays a fundamental role allowing the existence of a normalizable steady state by avoiding collapse of the pdf at the origin. When both kinds of noise occur simultaneously, the presence of multiplicative noise can not be formally avoided by a simple transformation of variables. The particular interplay between additive and multiplicative noises as well as that between deterministic and stochastic drifts can lead to the appearance of q-exponentials.

It is worthy to emphasize at this point that the stationary solutions of the present problem have the general form given by Eq. (14). The q-exponential pdf's represent a special case, which in turn includes the Boltzmann-Gibss pdf as an even more special one, corresponding to the standard thermal equilibrium. Furthermore, the q-exponential pdf's are unique in the sense that they optimize, under appropriate constraints, the entropy functional S_q (Eq. (10)). This entropic form is in turn unique in the sense that it is the only one which satisfies [23] a set of conditions naturally generalizing both the Shannon and the Kinchin axioms. Also, S_q is consistent with stability (or robustness) of the q-exponential pdf's in the sense described by Abe [24], whereas Renyi entropy and the normalized version of S_q are not.

Let us conclude by stressing that q-exponentials have also been observed in a variety of similar processes [5,18,25]. The mechanism leading to such distributions is expected to be present in systems with long–range memory, long-range interactions, fractal or hierarchical structures and similar scenarios.

In what concerns the model defined by Eqs. (3)-(5), it is clear that $f(u)$ and $g(u)$ are generically independent functions. However, it does occur that when they are connected through Eq. (7), the q-exponential form emerges naturally. It would no doubt be very interesting if we had a geometrical interpretation of this fact. Hints along this line would be welcome.

As illustrated in Fig. 3 (deterministic drift proportional to the stochastic one), the q-exponential is exact for $t \to \infty$ and it is an excellent approximation $\forall t$.

The best characterizations of the width of the pdf's clearly are those which remain finite under generic circumstances. In our case, this happens for $1/P(0,t)$, Δ, $\langle |u|^{2s} \rangle_q^{1/2s}$ or combinations such as $\sqrt{\Delta/P(0,t)}$. The latter two are preferable since they contain attributes of both body and tails of the pdf. For a mathematically convenient and generic characterization, clearly $\langle |u|^{2s} \rangle_q^{1/2s}$ is the most appropriate.

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FIGURE CAPTIONS

Fig. 1. Steady state pdfs for s=0.5 (a), 1 (b), and 2 (c), with $r = 2s - 1$, $A/\gamma = 1$ and different values of M/γ indicated in the figure. Notice that the alternative representation $\lambda P_s(u)$ vs. u/λ with $\lambda = (A/M)^{1/2s}$ would give profiles independent on the particular choice of A/γ.

Fig. 2. Different parameters characterizing the "width" of the distribution as a function of M/γ for s=0.5 (a), 1 (b), and 2 (c), with $r = 2s - 1$ and $A/\gamma = 1$. For $\gamma > 0$, normalizabity implies $M/\gamma < 1/(1-s)$ if $s < 1$; for $s \geq 1$, the physical region of M/γ extends up to infinity.

Fig. 3. Time evolution of the pdf for $r = s = 1$, $A = \gamma = 1$ and $M = 0.2$ at different times indicated in the figure, in different representations: linear (a), semi-$\ln_q$ with $q = \frac{\gamma + 3M}{\gamma + M}$ (b), and semi-$\ln_{q'}$, with time dependent q' (c). We used the ansatz $P = P_o(1 + (q' - 1)\beta u^2)^{1/(1-q')}$, with $P_o = \sqrt{\frac{\pi}{\beta(q'-1)} \frac{\Gamma(\frac{1}{q'-1} - \frac{1}{2})}{\Gamma(\frac{1}{q'-1})}}$. In the inset we present q' and β vs. t. In particular, $q(0) = 1$, $1/\beta(0) = 0$, $q(\infty) = 4/3$ and $1/\beta(\infty) = 5/3$.
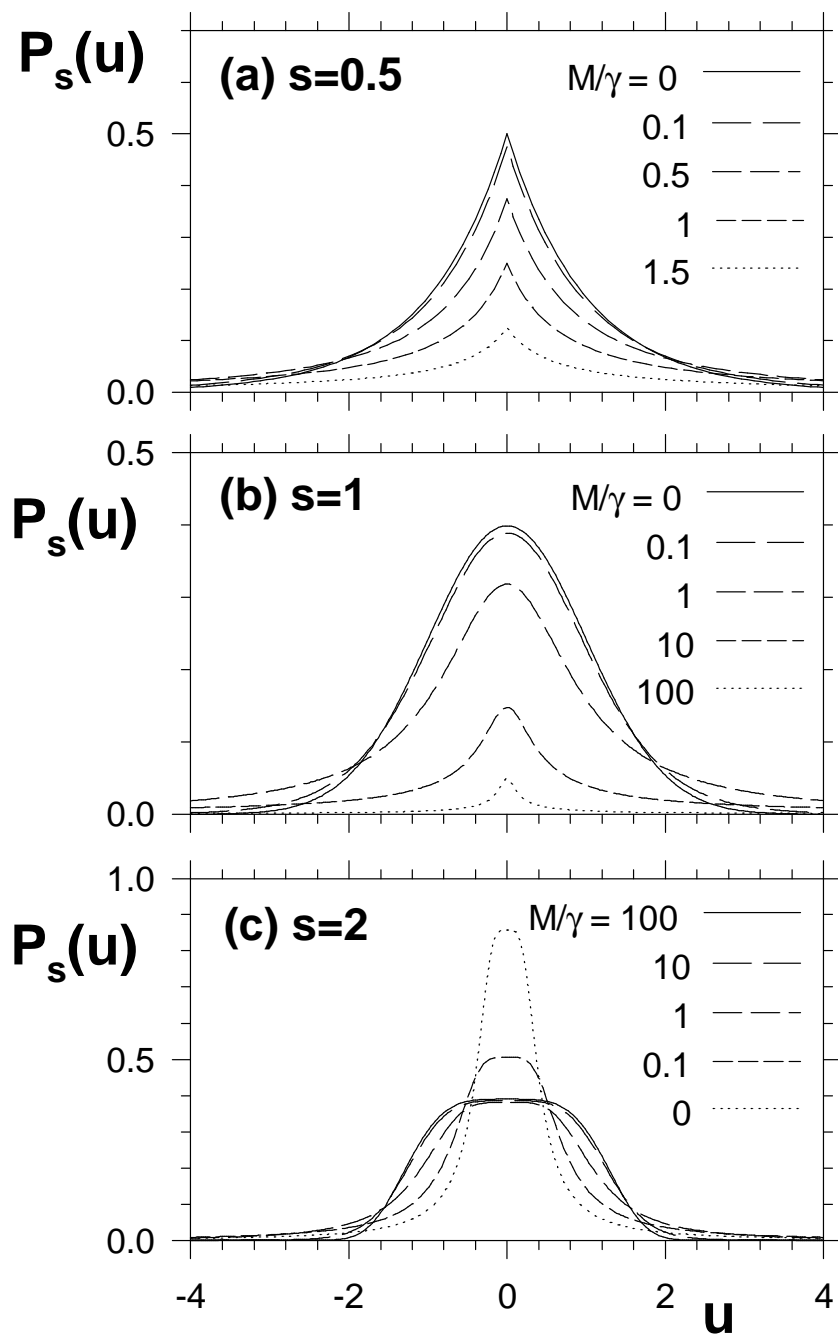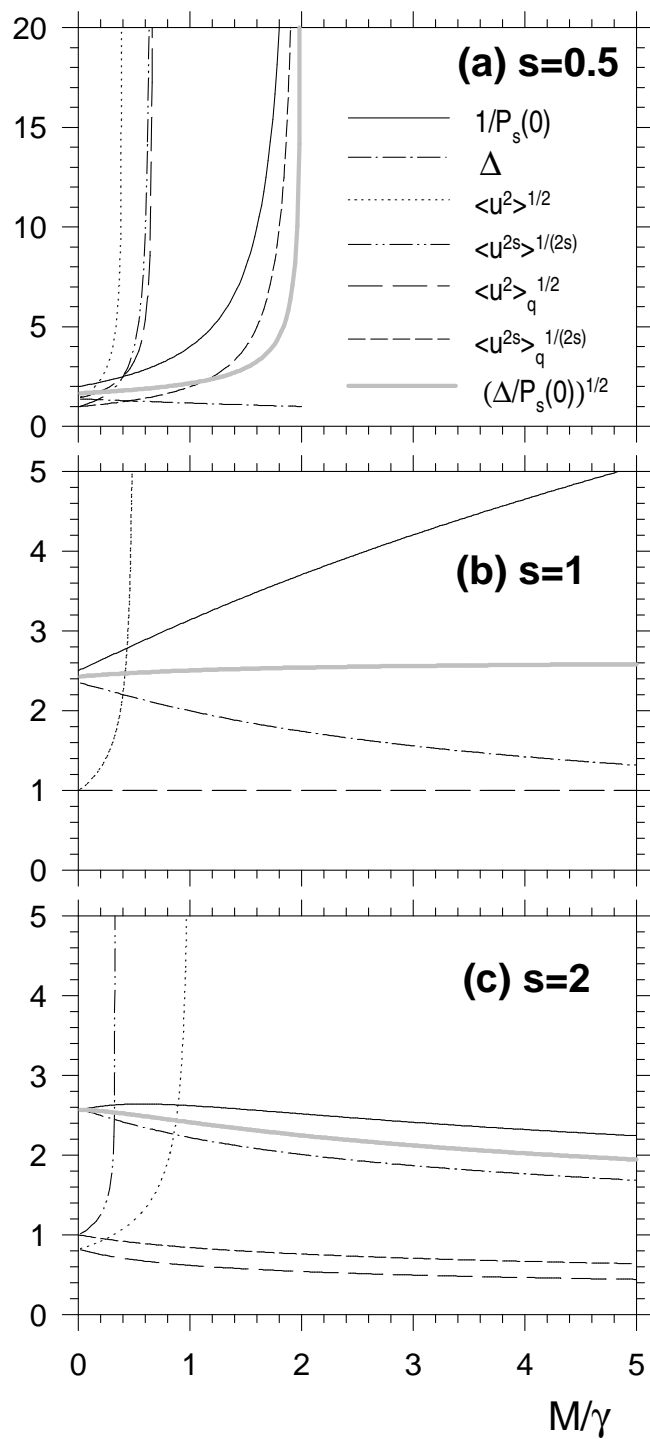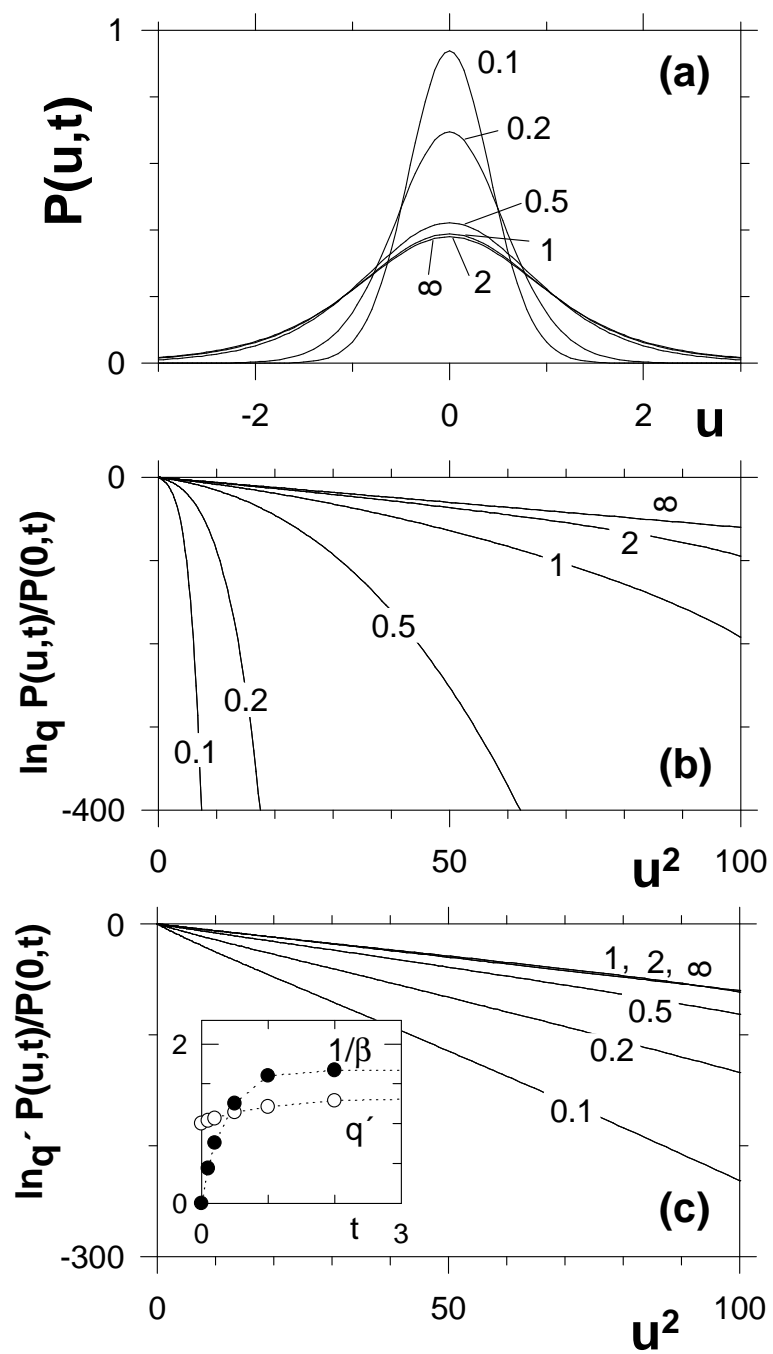
Figure 1

(a) s=0.5

- $1/P_s(0)$
- Δ
- $\langle u^2 \rangle^{1/2}$
- $\langle u^{2s} \rangle^{1/(2s)}$
- $\langle u^2 \rangle_q^{1/2}$
- $\langle u^{2s} \rangle_q^{1/(2s)}$
- $(\Delta/P_s(0))^{1/2}$

(b) s=1

(c) s=2

M/γ

Figure 2

Figure 3